The Fractional Fourier Transform and Its Applications

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Abstract

The Fractional Fourier transform (FrFT), as a generalization of the classical Fourier Transform, was introduced many years ago in mathematics literature. The original purpose of FrFT is to solve the differential equation in quantum mechanics. Optics problems can also be interpreted by FrFT. In fact, most of the applications of FrFT now are applications on optics. But there are still lots of unknowns to the signal processing community. Because of its simple and beautiful properties in Time-Frequency plane, we believe that many new applications are waiting to be proposed in signal processing.

In this paper, we will briefly introduce the FrFT and a number of its properties. Then we give one method to implement the FrFT in digital domain. This method to implement FrFT is based on Discrete Fourier Transform (DFT). Generally speaking, the possible applications of FT are also possible applications of FrFT. The possible applications in optics and signal processing are also included in Chapter 5.

1   Introduction

Fourier analysis is one of the most frequently used tools is signal processing and many other scientific fields. Besides the Fourier Transform (FT), time-frequency representations of signals, such as Wigner Distribution (WD), Short Time Fourier Transform (STFT), Wavelet Transform (WT) are also widely used in speech processing, image processing or quantum physics.

Many years ago, the generalization of the Fourier Transform, called Fractional Fourier Transform (FrFT), was first proposed in mathematics literature. Many new applications of Fractional Fourier Transform are found today. Although it is potentially useful, there seems to have remained largely unknown in signal processing

field. Recently, FrFT has independently discussed by lots of researchers. The purpose of this paper is threefold: First, to briefly introduce the Fractional Fourier Transform and its properties including the most important but simple interpretation as a rotation in the time-frequency plane. Second, derive the Discrete Fractional Fourier Transform and find the efficient ways to obtain the approximation of Continuous Fractional Fourier Transform. Third, I give some important applications, that is, now widely used in optics and signal processing.

2   Background

Because the Fractional Fourier Transform comes from the conventional Fourier Transform, we first review the Fourier Transform in this chapter.

2.1   Definitions of Fourier Transforms

The definitions of Fourier Transforms depend on the class of signals. We simply divide Fourier Transform into four categories:

   a)   Continuous-time aperiodic signal
   b)   Continuous-time periodic signal
   c)   Discrete-time aperiodic signal
   d)   Discrete-time periodic signal

The definitions in these categories are different but similar forms. Because in this paper we don't focus on conventional Fourier Transform, here we just list the definitions in Table 1, where the multi-dimensional Fourier Transforms are also defined in the similar form. There are many properties of Fourier Transform. But different signal class leads to a different form of properties, so we omit the properties of conventional Fourier Transform here.

3   Fractional Fourier Transform

3.1   Basic Concept of Fractional Transform

So far, we've seen the definitions of conventional Fourier Transform. Before formally defining the Fractional Fourier Transform, we want to know that "What is a fractional transform?" and "How can we make a transformation to be fractional?" First we see a transformation T, we can describe the transformation as following:

$$T\{f(x)\} = F(u) \tag{1}$$

where f and F are two functions with variables x and u respectively. As seen, we can say that F is a T transform of f. Now, another new transform can be defined as below:

$$T^{\alpha}\left\{f(x)\right\}=F_{\alpha}(u) \tag{2}$$

We call T^{α} here the "α -order fractional T transform" and the parameter α is called the "fractional order". This kind of transform is called "fractional transform". Which satisfy following constraints:

1. Boundary conditions:

$$T^{0}\left\{f(x)\right\}=f(u) \tag{3}$$

$$T^{1}\left\{f(x)\right\}=F(u) \tag{4}$$

Table 1 The definitions of Fourier Transform and its inverse for four different signal categories

Signal class	Definition of Fourier Transform and its inverse
Continuous-time aperodic signal	$$X(f)=\int_{-\infty}^{+\infty}x(t)e^{-j2\pi ft}dt$$ $$x(t)=\int_{-\infty}^{+\infty}X(f)e^{j2\pi ft}df$$
Continuous-time perodic signal (Fourier series expansion, FS)	$$S(kF)=\int_{0}^{Tp}s(t)e^{-j2\pi kFt}dt$$ $$s(t)=F\sum_{k=-\infty}^{+\infty}S(kF)e^{j2\pi kFt}$$
Discrete-time aperodic signal (Discrete-time Fourier Transform, DTFT)	$$S(f)=T\sum_{n=-\infty}^{+\infty}s(nT)e^{-j2\pi fnT}$$ $$s(nT)=\int_{0}^{Fp}S(f)e^{j2\pi fnT}df$$
Discrete -time perodic signal (Discrete Fourier Transform, DFT)	$$S(kF)=T\sum_{n=0}^{N-1}s(nT)e^{-j2\pi kFnT}$$ $$s(nT)=F\sum_{k=0}^{N-1}S(kF)e^{j2\pi kFnT}$$

2.   Additive property:

$$T^{\beta}\left\{T^{\alpha}\left\{f(x)\right\}\right\}=T^{\beta+\alpha}\left\{f(x)\right\} \tag{5}$$

Now, we can briefly derive the form of Fractional Fourier Transform. We use the eigenfunction of the Fourier Transform pairs to find the kernel of Fractional Fourier Transform.

3.2   Definition of Fractional Fourier Transform

The eigenvalues and eigenfunctions of the conventional Fourier Transform are well known. The two functions f and F are a Fourier Transform pair if:

$$F(v)=\int_{-\infty}^{+\infty}f(x)\exp(-i2\pi vx)dx \tag{6}$$

$$f(x)=\int_{-\infty}^{+\infty}F(v)\exp(i2\pi vx)dv \tag{7}$$

In the operator notation we can write $F=\Im\{f\}$ where $\Im$ denotes the conventional Fourier Transform. And we can easily find that $\Im^{2}\left\{f(x)\right\}=f(-x)$ and $\Im^{4}\left\{f(x)\right\}=f(x)$. The notation $\Im^{\alpha}$ means doing the operator $\Im$ for α times. Consider the equation

$$f''(x)+4\pi^{2}\left[(2n+1)/2\pi-x^{2}\right]f(x)=0 \tag{8}$$

By taking its Fourier Transform, we have

$$F''(v)+4\pi^{2}\left[(2n+1)/2\pi-v^{2}\right]F(v)=0 \tag{9}$$

We can find that the solutions of Eq. (9), known as Hermite-Gauss functions, are the eigenfunctions of the Fourier Transform operation. The normalized functions can form an orthonormal set, these functions are given by

$$\Psi_{n}(x)=\frac{2^{1/4}}{\sqrt{2^{n}n!}}H_{n}\left(\sqrt{2\pi}x\right)\exp\left(-\pi x^{2}\right) \tag{10}$$

for $n=0,1,2,\ldots$. These functions satisfy the eigenvalue equation

$$\Im\left[\Psi_{n}(x)\right]=\lambda_{n}\Psi_{n}(x) \tag{11}$$

4

where $\lambda_n = e^{-in\pi/2}$ are the eigenvalues of conventional Fourier Transform. Because the Hermite-Gaussian functions form a complete orthonormal set , we can than calculate the Fourier Transform by expressing it in terms of these eigenfunctions as following:

$$f(x) = \sum_{n=0}^{\infty} A_n \Psi_n(x) \tag{12}$$

$$A_n = \int_{-\infty}^{+\infty} \Psi_n(x) f(x) dx \tag{13}$$

$$\Im\{f(x)\} = \sum_{n=0}^{\infty} A_n e^{-in\pi/2} \Psi_n(x) \tag{14}$$

The αth order Fractional Fourier Transform shares the same eigenfunctions as the Fourier Transform, but its eigenvalues are the αth power of the eigenvalues of the ordinary Fourier Transform:

$$\Im^{\alpha}\{\Psi_n(x)\} = e^{-i\alpha n\pi/2} \Psi_n(x) \tag{15}$$

that is, the Fractional operator of order α may be defined through its effect on the eigenfunctions of the conventional Fourier operator.

If we define our operator to be linear, the fractional transform of an arbitrary function can be expressed as:

$$\{\Im^{\alpha}[f(x)]\}(x) = \sum_{n=0}^{\infty} A_n e^{-i\alpha n\pi/2} \Psi_n(x) \tag{16}$$

The definition can be cast in the form of a general linear transformation with kernel $B_{\alpha}(x, x')$ by insertion of Eq. (13) into Eq. (16):

$$\{\Im^{\alpha}[f(x)]\}(x) = \int_{-\infty}^{+\infty} B_{\alpha}(x, x') f(x') dx' \tag{17}$$

$$
\begin{aligned}
B_{\alpha}(x, x') &= \sum_{n=0}^{\infty} \lambda_n^{\alpha} \Psi_n(x) \Psi_n(x') \\
&= 2^{1/2} \exp\left[-\pi(x^2 + x'^2)\right] \\
&\quad \times \sum_{n=0}^{\infty} \frac{e^{-i\alpha n\pi/2}}{2^n n!} H_n\left(\sqrt{2\pi}x\right) H_n\left(\sqrt{2\pi}x'\right)
\end{aligned}
\tag{18}
$$

5

This can be reduced to a simpler form for $0 < |\phi| < \pi$:

$$B_\alpha(x,x') = \frac{\exp\left[-i\left(\pi\hat{\phi}/4 - \phi/2\right)\right]}{|\sin\phi|^{1/2}}$$

$$\times \exp\left[i\pi\left(x^2\cot\phi - 2xx'\csc\phi + x'^2\cot\phi\right)\right] \tag{19}$$

where $\phi = \alpha\dfrac{\pi}{2}$ and $\hat{\phi} = \text{sgn}(\sin\phi)$. We see that for $\alpha = 0$ and $\alpha = 2$, the

kernel reduces to $B_0(x,x') = \delta(x-x')$ and $B_2(x,x') = \delta(x+x')$. These

kernels correspond to the $0th$ and $2th$ order Fractional Fourier Transform and as

mentioned we saw the results are $f(x)$ and $f(-x)$. Some essential properties are

listed below

1. The Fractional Fourier Transform operator is linear.
2. The first-order transform $\mathfrak{I}^1$ corresponds to the conventional Fourier transform
    $\mathfrak{I}$ and the zeroth-order transform $\mathfrak{I}^0$ means doing no transform.
3. The fractional operator is additive, $\mathfrak{I}^\beta\mathfrak{I}^\alpha = \mathfrak{I}^{\beta+\alpha}$.

The kernel of the Fractional Fourier Transform can also be defined in the following

equation:

$$K_\phi(t,u) = \begin{cases} \sqrt{\dfrac{1-j\cot\phi}{2\pi}}\, e^{j\frac{t^2+u^2}{2}\cot\phi - jut\csc\phi} & \text{if } \phi \text{ is not a multiple of } \pi \\ \delta(t-u) & \text{if } \phi \text{ is a multiple of } 2\pi \\ \delta(t+u) & \text{if } \phi+\pi \text{ is a multiple of } 2\pi \end{cases} \tag{20}$$

And the Fractional Fourier Transform is defined by means of the transformation

kernel:

$$X_\phi(u) = \int_{-\infty}^{+\infty} x(t)K_\phi(t,u)\,dt$$

$$= \begin{cases} \sqrt{\dfrac{1-j\cot\phi}{2\pi}}\, e^{j\frac{u^2}{2}\cot\phi}\displaystyle\int_{-\infty}^{+\infty} x(t)e^{j\frac{t^2}{2}\cot\phi - jut\csc\phi}\,dt & \text{if } \phi \text{ is not a multiple of } \pi \\ \delta(t) & \text{if } \phi \text{ is a multiple of } 2\pi \\ \delta(-t) & \text{if } \phi+\pi \text{ is a multiple of } 2\pi \end{cases} \tag{21}$$

Table 2   Fractional Fourier Transform of some signals

Signal	FrFT with order ϕ
$\delta(t-\tau)$	$\sqrt{\dfrac{1-j\cot\phi}{2}}e^{j\frac{\tau^2+u^2}{2}\cot\phi}e^{-j\csc\phi\tau u}$
$e^{-j\left(at^2+bt+c\right)}$	$\sqrt{\dfrac{1-j\cot\phi}{j\cdot 2a-j\cot\phi}}e^{\frac{j}{2}\cdot\frac{2a\cot\phi-1}{\cot\phi-2a}u^2}e^{-j\frac{b\csc\phi}{\cot\phi-2a}}e^{-j\frac{b^2}{2(\cot\phi-2a)}-jc}$
1	$\sqrt{1+j\tan\phi}\cdot e^{-\frac{j}{2}u^2\tan\phi}$
$\cos(vt)$	$\sqrt{1+j\tan\phi}\cdot e^{-\frac{j}{2}\left(u^2+v^2\right)\tan\phi}\cos(uv\sec\phi)$
$\sin(vt)$	$\sqrt{1+j\tan\phi}\cdot e^{-\frac{j}{2}\left(u^2+v^2\right)\tan\phi}\sin(uv\sec\phi)$

We can see some facts about definition in Eq. (21). When $\phi=0$, the transformation

of a function is itself. When $\phi=\dfrac{\pi}{2}$, the transformation becomes conventional

Fourier Transform. These satisfy the boundary condition we saw in Eq. (3) and Eq. (4). The additive condition can be proved by simply applying two different kernels in the transformation. In Table (2), we give FrFT of some common signals. And we summarize the properties of Fractional Fourier Transform is listed in Table (3).

3.3   Linear Canonical Transform

We have seen that Fractional Fourier Transform is the general form of conventional Fourier Transform, whereas there is a more general form these transform which is called the "*Linear Canonical Transform (LCT)*". The Linear Canonical Transform is defined as:

$$O_F^{(a,b,c,d)}\left\{g(t)\right\}=G_{(a,b,c,d)}(u)$$

$$=\sqrt{\frac{1}{j2\pi b}}e^{\frac{j}{2}\frac{d}{b}u^2}\int_{-\infty}^{+\infty}e^{-\frac{j}{b}ut}e^{\frac{j}{2}\frac{a}{b}t^2}g(t)dt \qquad \text{for } b\neq 0 \tag{22}$$

$$O_F^{(a,b,c,d)}\left\{g(t)\right\}=\sqrt{d}\,e^{\frac{j}{2}cd\,u^2}g(du) \qquad \text{for } b=0 \tag{23}$$

The Linear Canonical Transform is a further generalization of Fractional Fourier

Transform. When $(a,b,c,d)=(\cos\alpha,\sin\alpha,-\sin\alpha,\cos\alpha)$, the LCT becomes

FrFT. And the parameters (a,b,c,d) satisfy $ad-bc=1$. The LCT also has following additive property:

$$O_F^{(a_2,b_2,c_2,d_d)}\left\{O_F^{(a_1,b_1,c_1,d_1)}\left[g(t)\right]\right\}=O_F^{(e,f,g,h)}\left\{g(t)\right\} \tag{24}$$

where $\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$.

In section 5, we can see that LCT can describe optical systems contain arbitrary sections of quadratic graded-index media or even arbitrary thin filters and so on. In this paper, we focus on the Fractional Fourier Transform so the details of LCT are not mentioned here.

3.4 Relations to Other Transformations

There are many relations from Fractional Fourier Transform to many other transformations such as Wigner Distribution (WD) and Gabor Transform (GT). In this

Table 3 Some properties of Fractional Fourier Transform

Aperiodic signal	FrFT with angle ϕ	Description
$ax(t)+by(t)$	$aX_\phi(u)+bY_\phi(u)$	Linearity
$x(t-T)$	$e^{j\frac{T^2}{2}\cot\phi}e^{-juT\csc\phi}X_\phi(u-T\cos\phi)$	Time shift
$e^{j2\pi Ft}x(t)$	$e^{-j\frac{v^2\sin\phi\cos\phi}{2}+juv\cos\phi}X_\phi(u-v\sin\phi)$	Modulation
$\dfrac{d}{dt}x(t)$	$\cos\phi\dfrac{d}{du}X_\phi(u)+ju\sin\phi X_\phi(u)$	Derivative
$\displaystyle\int_{-\infty}^{t}x(t)dt$	$\sec\phi\cdot e^{-\frac{j}{2}u^2\tan\phi}\displaystyle\int_{-\infty}^{u}e^{\frac{j}{2}v^2\tan\phi}X_\phi(v)dv$	Integration
$tx(t)$	$u\cos\phi X_\phi(u)+j\sin\phi\dfrac{d}{du}X_\phi(u)$	Time multiplication
$\displaystyle\int_{-\infty}^{+\infty}x(t)y^*(t)dt=\int_{-\infty}^{+\infty}X_\phi(u)Y_\phi^*(u)du$		Parseval relation

section, we introduce some relations between them. These relations are quite important because many applications are based on them.

3.4.1　Relation to Wigner Distribution

The direct and simple relationship of the Fractional Fourier Transform to the Wigner Distribution (WD) as well as to certain other phase-space distributions is perhaps its most important and elegant property.

This property states that performing the αth order Fractional Fourier Transform operation corresponds to rotating the Wigner Distribution by an angle $\phi = \alpha\dfrac{\pi}{2}$ in the clockwise direction. The Wigner Distribution of a function is defined as:

$$W\{f(x)\} = W(x,v)$$
$$= \int_{-\infty}^{+\infty} f(x + x'/2) f^*(x - x'/2) \exp(-j2\pi vx') dx' \tag{25}$$

$W(x,v)$ can also be expressed in terms of $F(v)$, or indeed as a function of any fractional transform of $f(x)$. There are some properties that are most relevant:

$$|f(x)|^2 = \int W(x,v) dv \tag{26}$$

$$|F(v)|^2 = \int W(x,v) dx \tag{27}$$

$$\text{Total energy } E\big[f(x)\big] = \int W(x,v) dx dv \tag{28}$$

Roughly speaking, $W(x,v)$ can be interpreted as a function that indicates the distribution of the signal energy over space and frequency. Now, if $W_f(x,v)$ denotes the Wigner Distribution of $f(x)$, then the Wigner Distribution of the αth order Fractional Fourier Transform of $f(x)$, denoted by $W_{f_\alpha}(x,v)$ is given by:

$$W_{f_\alpha}(x,v) = W_f(x\cos\phi - v\sin\phi, x\sin\phi + v\cos\phi) \tag{29}$$

Obviously, the Wigner Distribution of the αth order Fractional Fourier Transform of $f(x)$ is obtained from $W_f(x,v)$ by rotating it clockwise by an angle ϕ.

If we define the rotation operation R_ϕ for two dimensional functions, corresponding to a counterclockwise rotation by ϕ. Then Eq. (29) can be expressed as:

$$W[f_\alpha] = R_{-\phi} W[f_0] \tag{30}$$

Because Fractional Fourier Transform and the rotation operators are additive with respect to their parameters, we can easily generalize Eq. (30) to:

$$W\left[f_{\alpha_2}\right] = R_{(-\phi_2 + \phi_1)} W\left[f_{\alpha_1}\right] \tag{31}$$

Now, we see Eq. (26) and Eq. (27), these functions can rewrite for the $f_\alpha(v)$:

$$\int W_{f_\alpha}(x,v)dx = \left|f_\alpha(v)\right|^2 \tag{32}$$

Eq. (32) can again be rewritten by an operator $\Re_\phi$, which is the *Radon transform* evaluated at the angle ϕ. The Radon transform of a two-dimensional function is its projection on an axis making angle ϕ with the x_0 axis. Eq. (32) rewritten by Radon transform is:

$$\Re_\phi\{W[f]\} = \left|f_\alpha(v)\right|^2 \tag{33}$$

3.4.2   Relation to Chirp Transform

We begin this part by considering the following functions and their corresponding Wigner distributions:

$$f(x) = \exp(j2\pi v_c x) \quad W(x,v) = \delta(v - v_c) \tag{34}$$

$$f(x) = \delta(x - c) \quad W(x,v) = \delta(x - c) \tag{35}$$

$$\begin{aligned} f(x) &= \exp\left[j2\pi\left(b_2 x^2/2 + b_1 x + b_0\right)\right] \\ W(x,v) &= \delta(b_2 x + b_1 - v) \end{aligned} \tag{36}$$

The first of these results shows that the Wigner Distribution of a pure harmonic is a line delta along $v - v_c$. The second one shows that the Wigner Distribution of a delta function remains delta function along $x = c$. The third one is chirp function, its Wigner Distribution is a line delta making an angle $\phi = \tan^{-1} b_2$ with the x axis.

10
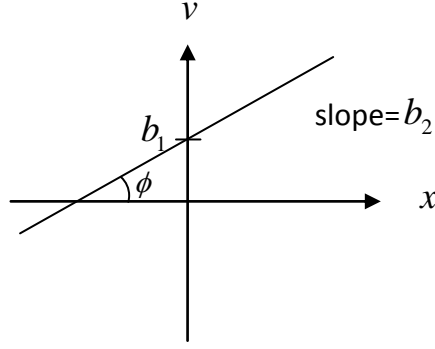
Fig. 1   Wigner distribution of a chirp function

This is shown in **Fig. 1**.

Recall that in 3.4.1 we saw the effect of Fractional Fourier Transform is to rotate the Wigner Distribution of a function. Thus we can suspect that a chirp function is the $\alpha = 0$ domain representation of pure harmonics or delta function in other fractional Fourier domains.

Now, by using the kernel in Eq. (19) we have the Fractional Fourier Transform of a shift delta function $\delta(x_0 - x_{0c})$ is

$$f_\alpha(x_\alpha) = \frac{\exp\left[-j\left(\pi\hat{\phi}/4 - \phi/2\right)\right]}{|\sin\phi|^{1/2}} \tag{37}$$
$$\times \exp\left[j\pi\left(x_\alpha^2 \cot\phi - 2x_\alpha x_{0c}\csc\phi + x_{0c}^2\cot\phi\right)\right]$$

Here we use x_{0c} denotes a constant to the x_0 axis. For $\alpha = 1$ $(\phi = \pi/2)$, this reduces to $\exp(-j2\pi v x_{0c})$, as expected. $f_\alpha(x_\alpha)$ can be considered an alternative representation of $f_0(x_0)$ in the αth fractional domain. Because $f_\alpha(x_\alpha)$ is the αth order Fractional Fourier Transform of $f_0(x_0)$, from last section, the Wigner Distribution of $f_\alpha(x_\alpha)$ must be a rotated version of which of $f_0(x_0)$. This is shown as follows:

$$W(x_0, x_1) = \delta(x_0\cos\phi - x_1\sin\phi - x_{0c}) \tag{38}$$

Which we verify that the Wigner Distribution of $\delta(x_0 - x_{0c})$ is also $\delta(x_0 - x_{0c})$

11

rotated by $-\phi$. So we can say that a delta function in the αth domain, $\delta(x_\alpha - x_{0c})$, is a chirp function in the α'th domain.

Now, consider the identity:

$$f(x) = f_0(x_0) = \int f_0(x')\delta(x_0 - x')dx' \tag{39}$$

and we do the αth order Fractional Fourier Transform on both sides, then we obtain:

$$f_\alpha(x_\alpha) = \int f_0(x')\mathfrak{I}^\alpha\{\delta(x_0 - x')\}dx' \tag{40}$$

where $\mathfrak{I}^\alpha\{\delta(x_0 - x')\}$ is already given in Eq. (37). For special case when $\alpha = 1$,

that is, the Fourier Transform of $\delta(x_0 - x')$ is the pure harmonic

$\exp(-j2\pi x_1 x') = \exp(-j2\pi f x')$, which is the kernel of conventional Fourier

Transform.

In fact, the representation of a signal in the αth domain is what we call the αth order Fractional Fourier Transform of the signal in $\alpha = 0$ (time or space) domain. But generally, if the representation of a signal in α'th domain is known, we can find

its representation in the αth by taking $(\alpha - \alpha')$th order Fractional Fourier

Transform.

As the same in Eq. (39), we can also represent a signal in the αth fractional domain:

$$f_\alpha(x_\alpha) = \int f_\alpha(x')\delta(x_\alpha - x')dx' \tag{41}$$

or we can represent it as a superposition of harmonics in the $(\alpha + 1)$th domain:

$$f_\alpha(x_\alpha) = \int F_\alpha(v_\alpha)\exp(j2\pi v_\alpha x_\alpha)dv_\alpha \tag{42}$$

where $F_\alpha = f_{\alpha+1}$, $v_\alpha = x_{\alpha+1}$ and $v = v_0 = x_1$. More generally, Eq. (42) can be

rewritten as the superposition of chirp function in other α'th domain:

$$f_\alpha(x_\alpha) = \int f_{\alpha'}(x_{\alpha'})B_{\alpha-\alpha'}(x_\alpha, x_{\alpha'})dx_{\alpha'} \tag{43}$$

where $B_{\alpha-\alpha'}(x_\alpha, x_{\alpha'})$ is the kernel of $(\alpha - \alpha')$th order Fractional Fourier

Transform. This is equivalent to finding the projection of $f_{\alpha'}$ in the α'th domain onto the basis function. Note that the representation of these basis functions in the original α'th domain is chirp function.

3.4.3   Relation to Gabor Transform

Gabor Transform is one of the time-frequency analysis tools. It is a special case of the Short-Time Fourier Transform (STFT), where the window function it uses is the Gaussian function. The Gabor Transform can thus be written by

$$G_f(t,\omega)=\sqrt{1/2\pi}\int_{-\infty}^{\infty}e^{-\frac{(\tau-t)^2}{2}}e^{-j\omega(\tau-\frac{t}{2})}f(\tau)d\tau \tag{44}$$

If $F_\alpha(u)$ is the αth order Fractional Fourier Transform of $f(x)$, $G_f(t,\omega)$ is the Gabor Transform of $f(x)$ and $G_{F_\alpha}(u,v)$ is the Gabor Transform of $F_\alpha(u)$.

Then it can be proved that $G_f(t,\omega)$ and $G_{F_\alpha}(u,v)$ has the following relation:

$$G_{F_\alpha}(u,v)=G_f(u\cos\alpha-v\sin\alpha,u\sin\alpha+v\cos\alpha) \tag{45}$$

That is, we can find that, like the Wigner Distribution, the Fractional Fourier Transform of parameter α is equivalent to rotating the Gabor Transform in the clockwise direction with angle α. But why we use Gabor Transform instead of using Wigner Distribution. That is because the Gabor Transform is a linear operator and we

(a)GT of $s(t)$ (b)GT of $r(t)$ (c)GT of $f(t)$ (d)WD of $f(t)$
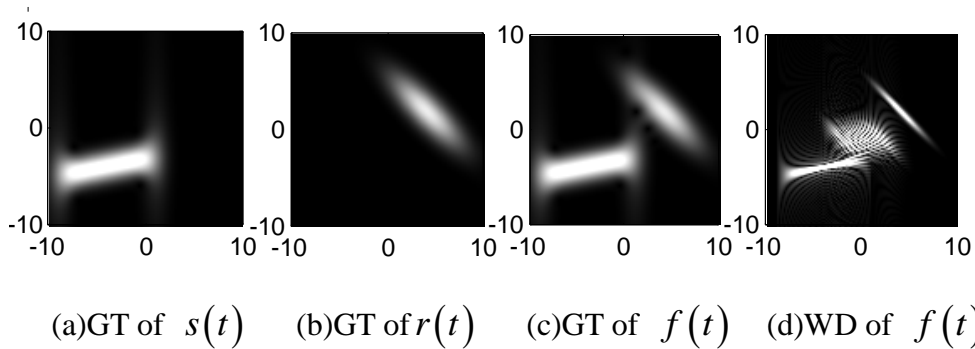
Fig. 2   The Gabor transforms (GTs) and the Wigner distribution (WDs) of $s(t)$, $r(t)$, and $f(t)=s(t)+r(t)$. Note that the WDF has the "cross-term problem" but not the GT.

need not to calculate the auto-correlation function $f(x+x'/2)f^*(x-x'/2)$, the cross-term problem of Wigner Distribution can thus be avoided. That is, if $f(t)=s(t)+r(t)$, and $G_f(t,\omega)$, $G_s(t,\omega)$ and $G_r(t,\omega)$ are their Gabor Transforms, then

$$G_f(t,\omega)=G_s(t,\omega)+G_r(t,\omega) \tag{46}$$

In **Fig. 2**, we give an example of the Gabor Transform and the Wigner Distribution of a signal $f(t)$. Where

$$s(t)=\exp\left(jt^2/10-j3t\right) \text{ for } -9\le t\le 1, \ s(t)=0 \text{ otherwise,}$$
$$r(t)=\exp\left(-jt^2/2+j6t\right)\exp\left[-(t-4)^2/10\right] \tag{47}$$
$$f(t)=s(t)+r(t)$$

We can see that the cross-term problem can be avoided if we use Gabor Transform instead of Wigner Distribution, see **Fig. 2**(c)(d).

Another advantage of Gabor Transform is that the computation time of the Gabor Transform will be much less than Wigner Distribution. For Wigner Distribution, see Eq. (25), we should compute the integral in the range $\left(-\infty,+\infty\right)$. But for Gabor Transform, we notice that there is a $e^{-\frac{(\tau-t)^2}{2}}$ term in the integral. From the fact that

$$\exp\left(-x^2/2\right)<0.0001 \quad \text{when } |x|>4.2919 \tag{48}$$

than we can approximate the Gabor Transform with the following equation

$$G_f(t,\omega)\approx\frac{1}{\sqrt{2\pi}}\int_{t-4.2919}^{t+4.2919}e^{-\frac{(\tau-t)^2}{2}}e^{-j\omega(\tau-\frac{t}{2})}f(\tau)d\tau \tag{49}$$

It is obvious that the computation range of Eq. (49) is much smaller that the Wigner Distribution one in Eq. (25).

But there is a drawback of Gabor Transform, that is, the resolution of the Gabor Transform of the signal is worse than the Wigner Distribution one. This can be found in **Fig. 2**(c)(d). The details here are omitted.

In order to combine the advantage of these two kinds of signal representation, that is, Wigner Distribution has higher clarity and Gabor Transform can avoid cross-term problem, S. C. Pei and J. J. Ding proposed a new transform called the *Gabor-Wigner*

Transform (GWT). The question now becomes how to combine these two transforms? We define a new time-frequency transform $C_f(t,\omega)$ called the Gabor-Wigner Transform (GWT) that has the following relation with the Gabor Transform $G_f(t,\omega)$ and the Wigner Distribution $W_f(t,\omega)$

$$C_f(t,\omega) = p\big(G_f(t,\omega), W_f(t,\omega)\big) \tag{50}$$

where $p(x,y)$ is any function with two variables. It can be proved that $C_f(t,\omega)$ also has the rotation relation with the Fractional Fourier Transform. By choosing appropriate $p(x,y)$, we can achieve our goals of combining the advantages of Gabor Transform and Wigner Distribution. In **Fig. 3**, we give examples of choosing different $p(x,y)$ of Gabor Wigner Transform to perform on the signals we have used in Eq. (47). The Gabor Wigner Transform we choose are

in **Fig. 3**(a) $$C_f(t,\omega) = G_f(t,\omega)W_f(t,\omega) \tag{51}$$

in **Fig. 3**(b) $$C_f(t,\omega) = \min\left\{ \left|G_f(t,\omega)\right|^2, W_f(t,\omega) \right\} \tag{52}$$

in **Fig. 3**(c) $$C_f(t,\omega) = W_f(t,\omega) \cdot \left\{ \left|G_f(t,\omega)\right| > 0.25 \right\} \tag{53}$$

in **Fig. 3**(d) $$C_f(t,\omega) = G_f^{2.6}(t,\omega)W_f^{0.6}(t,\omega) \tag{54}$$
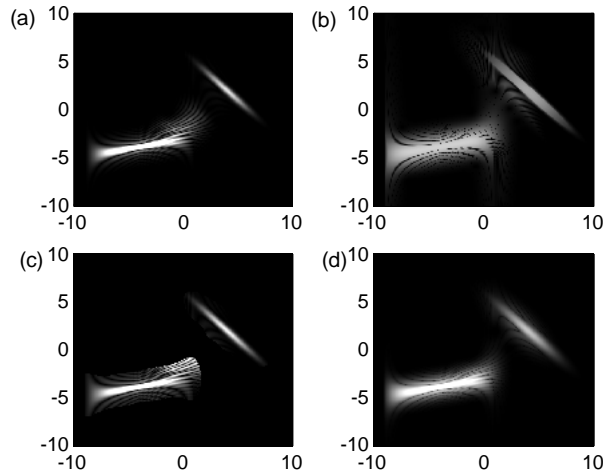
Fig. 3 the Gabor-Wigner Transform of $f(t)$, which is defined in Eq. (47), with

different choice of $p(x,y)$

We see that by choosing appropriate Gabor-Wigner Transform, we can have both high clarity and avoid the cross-term problem.

3.4.4   Relation to Wavelet Transform

The kernels of Fractional Fourier Transform corresponding to different values of α can be regarded as a wavelet family. See the Eq. (19), by the change of variable $y = x_\alpha \sec\phi$, we can write the Fractional Fourier Transform of function $f(x)$ as:

$$g(y) = f_\alpha\left(\frac{y}{\sec\phi}\right) = C(\phi)\exp\left(-j\pi y^2 \sin^2\phi\right)$$
$$\times \int \exp\left[j\pi\left(\frac{y - x'}{\tan^{1/2}\phi}\right)^2\right] f(x')dx'$$

(55)

We can take $\tan^{1/2}\phi$ as the scale parameter. And the above equation is the wavelet transform in which the wavelet family is obtained from the quadratic phase function $w(x) = \exp\left(j\pi x^2\right)$ by scaling the coordinate and the amplitude by $\tan^{1/2}\phi$ and $C(\phi)$, respectively.

This has recently been shown that the formulation of optical diffraction can be expressed in a similar wavelet framework. In Chapter 5, we will discuss the filtering at different Fractional Fourier domains. These operations can also be interpreted as filtering at the corresponding wavelet transform domain.

3.4.5   Relation to Random Process

In this section, we simply derive the relation between Fractional Fourier Transform and random process. First, we discuss the relation to stationary random process. Suppose that $g(t)$ is a stationary random process, and $G_\alpha(u)$ is the Fractional Fourier Transform of $g(t)$. Then we can calculate the autocorrelation function of $G_\alpha(u)$ by applying Eq. (21):

$$R_{G_\alpha}(u,\tau) = E\left[G_\alpha(u+\tau/2)G_\alpha^*(u-\tau/2)\right]$$

$$= \sqrt{\frac{1+\cot^2\alpha}{4\pi^2}} e^{\frac{j}{2}\cot\alpha\left[\left(u+\frac{\tau}{2}\right)^2 - \left(u-\frac{\tau}{2}\right)^2\right]}$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-j\left(u+\frac{\tau}{2}\right)t\csc\alpha} \cdot e^{j\left(u-\frac{\tau}{2}\right)t_1\csc\alpha} e^{\frac{j}{2}(t^2-t_1^2)\cot\alpha} E\left[g(t)g^*(t_1)\right]dtdt_1 \tag{56}$$

where $E\left[g(t)g^*(t_1)\right] = R_g(t-t_1)$. And by variable changing, Eq. (56) can be rewritten as

$$R_{G_\alpha}(u,\tau) = \frac{e^{ju\tau\cot\alpha}}{|\cos\alpha|} e^{-j\frac{u\tau}{\sin\alpha\cos\alpha}} R_g\left(\frac{\tau}{\cos\alpha}\right) \tag{57}$$

Although we can see that in Eq. (57), $R_{G_\alpha}(u,\tau)$ is not stationary. But the amplitude of $R_{G_\alpha}(u,\tau)$ is $|\sec\alpha| \cdot |R_g(\tau\sec\alpha)|$. It is independent of u. So we can say that $G_\alpha(u)$ is nearly stationary. Moreover, if $g(t)$ is real, since $R_G(\tau)$ is also real, we can thus conclude that

$$\arg\left[R_{G_\alpha}(u,\tau)\right] = -u\tau\tan\alpha \tag{58}$$

So we can use the phase of $R_{G_\alpha}(u,\tau)$ to estimate the parameter α of the Fractional Fourier Transform.

Now we turn back to Eq. (57), note that this equation is only for $\cos\alpha \neq 0$. For the case when $\alpha = (K+1/2)\pi$, that is, $\cos\alpha = 0$ Eq. (57) cannot be applied. So we get back to Eq. (56) which we applied that $\csc\alpha = 1/\sin\alpha = \pm1$. Then we have

$$R_{G_\alpha}(u,\tau) = \delta(\tau)S_g(u), \quad \text{when } \alpha = (2H+1/2)\pi \tag{59}$$

$$R_{G_\alpha}(u,\tau) = \delta(\tau)S_g(-u), \quad \text{when } \alpha = (2H+3/2)\pi \tag{60}$$

where H is some integer and $S_g(u)$ is the PSD of $g(t)$. The definition of power spectral density of a signal is

$$S_g(t,\omega) = \sqrt{2\pi} FT_{\tau\to\omega}\left[R_g(t,\tau)\right] = \int_{-\infty}^{+\infty} R_g(t,\tau)e^{-j\omega\tau}d\tau \qquad (61)$$

By substituting Eq. (57)(59)(60) into Eq. (61), we obtain $S_g(t,\omega)$ for $\cos\alpha \neq 0$ and $\cos\alpha = 0$ case:

$$S_{G_\alpha}(u,v) = S_g(u\sin\alpha + v\cos\alpha), \quad \text{when } \cos\alpha \neq 0 \qquad (62)$$

$$S_{G_\alpha}(u,v) = S_g(\pm u), \quad \text{when } \cos\alpha = 0 \qquad (63)$$

Thus, $S_{G_\alpha}(u,v)$ is a scaling and shifting version of $S_g(u,v)$ and the amount of shifting grows with u. Furthermore, Eq. (63) can be regarded as a special case of Eq. (62).

4 Discrete Fractional Fourier Transform

In last chapter, we saw the definition and properties of continuous Fractional Fourier Transform. Although the continuous Fractional Fourier Transform can be implemented by optical system, but it still need more convenient method to calculate it. In this chapter, we introduce a method to implement Fractional Fourier Transform. This method finds the discrete Fractional Fourier Transform by eigen-decompose the transform matrix of discrete Fourier Transform.

First, we see the Discrete Fourier Transform (DFT), which is the discrete version of Fourier Transform. The N-point Discrete Fourier Transform pair is defined as

$$X(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{nk}{N}}, \quad k = 0,1,\ldots,N-1 \qquad (64)$$

$$x(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} X(k)e^{j2\pi\frac{nk}{N}}, \quad n = 0,1,\ldots,N-1 \qquad (65)$$

where $\dfrac{1}{\sqrt{N}}$ is just a normalization factor, it makes both the DFT and IDFT unitary.

The N-point Discrete Fourier Transform in Eq. (64) can be written in a matrix form:

$$\mathbf{F}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1} & e^{-j\frac{2\pi}{N}2} & \cdots & \cdots & e^{-j\frac{2\pi}{N}(N-1)} \\ 1 & e^{-j\frac{2\pi}{N}2} & e^{-j\frac{2\pi}{N}4} & \cdots & \cdots & e^{-j\frac{2\pi}{N}2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)} & e^{-j\frac{2\pi}{N}(N-1)2} & \cdots & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix} \quad (66)$$

According to Eq. (64)(66), the N-point Discrete Fourier Transform can be written as

$$\mathbf{X_F} = \mathbf{F}_N \mathbf{x} \tag{67}$$

where $\mathbf{x}$ and $\mathbf{X_F}$ are both $N \times 1$ column vectors and we call $\mathbf{x}$ and $\mathbf{X_F}$ N-point Discrete Fourier Transform pairs. From above formula, if $\mathbf{F}_N$ is diagonalizable, that is, we can decompose $\mathbf{F}_N$ as

$$\mathbf{F}_N = \mathbf{U}\mathbf{D}\mathbf{U}^T \tag{68}$$

where $\mathbf{D}$ is the diagonal matrix consists of eigenvalues of $\mathbf{F}_N$ and $\mathbf{U}$ is the orthogonal matrix. In Eq. (66), it is obvious that $\mathbf{F}_N$ is a symmetric matrix. From the matrix theory, we know that a symmetric matrix is always orthogonally diagonalizable. Then, from the same idea in Eq. (15), we can calculate the Fractional Transform of $\mathbf{F}_N$ by

$$\mathbf{F}_N^\alpha = \mathbf{U}\mathbf{D}^\alpha \mathbf{U}^T \tag{69}$$

The eigenvalues of Discrete Fourier Transform are $\{+1, -1, +j, -j\}$ and multiplicities, that is, the repeat number of the eigenvalues depends on the remainder of $N \bmod 4$, this is listed in **Table 4**.

Table 4 eigenvalue multiplicity of DFT matrix

N	Mul. Of $+1$	Mul. Of -1	Mul. Of $+j$	Mul. Of $-j$
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	$m+1$	m	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

If we let $\omega = 2\pi/N$, and let matrix $\mathbf{S}$

$$\mathbf{S} = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 2\cos(\omega) & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\cos(2\omega) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos[(N-2)\omega] & 1 \\ 1 & 0 & 0 & \cdots & 1 & 2\cos[(N-1)\omega] \end{bmatrix} \quad (70)$$

It can be easily shown that $\mathbf{FS} = \mathbf{SF}$. Because $\mathbf{S}$, with distinct eigenvalues, commutes with $\mathbf{F}$, the eigenvectors of $\mathbf{S}$ will be the desired set of eigenvectors of $\mathbf{F}$. Note that $\mathbf{S}$ is a real and symmetric matrix, so its eigenvectors will be real and orthogonal. Now, we get back to Eq. (69). Because eigenvalues of $\mathbf{F}$ are $\{+1,-1,+j,-j\}$, it can be written by

$$\mathbf{F}_N^\alpha = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 & \mathbf{U}_3 & \mathbf{U}_4 \end{bmatrix} \begin{bmatrix} (\mathbf{I}_1)^\alpha & 0 & 0 & 0 \\ 0 & (-\mathbf{I}_2)^\alpha & 0 & 0 \\ 0 & 0 & (-j\mathbf{I}_3)^\alpha & 0 \\ 0 & 0 & 0 & (j\mathbf{I}_4)^\alpha \end{bmatrix} \begin{bmatrix} \mathbf{U}_1^T \\ \mathbf{U}_2^T \\ \mathbf{U}_3^T \\ \mathbf{U}_4^T \end{bmatrix} \quad (71)$$

where α is the order of discrete Fractional Fourier Transform and U_i are given by

1) $\mathbf{U}_1$ is constructed by the eigenvectors $\mathbf{v}$ of matrix $\mathbf{S}$ which satisfy $\mathbf{Fv} = \mathbf{v}$

2) $\mathbf{U}_2$ is constructed by the eigenvectors $\mathbf{v}$ of matrix $\mathbf{S}$ which satisfy $\mathbf{Fv} = -\mathbf{v}$

3) $\mathbf{U}_3$ is constructed by the eigenvectors $\mathbf{v}$ of matrix $\mathbf{S}$ which satisfy $\mathbf{Fv} = -j\mathbf{v}$

4) $\mathbf{U}_4$ is constructed by the eigenvectors $\mathbf{v}$ of matrix $\mathbf{S}$ which satisfy $\mathbf{Fv} = j\mathbf{v}$

5   Applications of Fractional Fourier Transform

A lot of applications of Fractional Fourier Transform have been proposed recently. Although most of them are the application of the Fractional Fourier Transform to optical problems, there still have many useful results for signal processing region. In fact, applications of Fourier Transform may be applications of Fractional Fourier Transform. In this chapter, I will introduce some applications of Fractional Fourier Transform such as applications to optical system, applications for filter design, applications for noise removal and so on.

5.1   Optics Analysis and its Implementation by Fractional Fourier Transform

Both Fractional Fourier Transform and Linear Canonical Transform can be used for optical system analysis, but when we want to analysis combination of 2 or more optical systems, it is more convenient to use Linear Canonical Transform. When we use Linear Canonical Transform, we can use matrices multiplications to analyze the combination of optical systems. However, if we use Fractional Fourier Transform, we need to do the integral calculation.

5.1.1   Using FrFT/LCT to Represent Optical Components

a)   Propagation through the cylinder lens with focus length f

Suppose the monochromatic light, with wavelength λ and it has the field distribution $U_i(x)$, enters to a cylinder lens with focus length f, thickness Δ, and refractive index of η. Then from the paraxial approximation, the output will have the distribution as $U_o(x)$ as

$$U_o(x) = e^{j2\pi\eta\Delta/\lambda} \cdot e^{-j\frac{\pi}{\lambda \cdot f}x^2} \cdot U_i(x) \tag{72}$$

If we ignore the constant phase, we find it just corresponds to the Linear Canonical Transform with the parameters

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2\pi/\lambda f & 1 \end{bmatrix} \tag{73}$$

b)   Propagation through the free space (Fresnel Transform) with length z

As the same assumption in a), the relation between input and output distribution when light propagates through the free space with length z is

$$U_o(s) = \frac{e^{j2\pi z/\lambda}}{j\lambda z} \cdot \int_{-\infty}^{+\infty} e^{j\frac{\pi}{\lambda z}(s-x)^2} \cdot U_i(x) \cdot dx \tag{74}$$

This is called the Fresnel Transform. Then, compare with Eq. (22), we find that if the constant phase is ignored, it corresponds to the Linear Canonical Transform with parameters as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z/2\pi \\ 0 & 1 \end{bmatrix} \tag{75}$$

Besides a) and b), there are also other optical propagation can be represented by Linear Canonical Transform.

5.1.2 Using FrFT/LCT to Represent the Optical Systems

According to 5.1.1, since Linear Canonical Transform can represent the 2 optical operations described above, then we can use Linear Canonical Transform to represent the optical systems composed of these two operations. We can follow the steps below:

1. For each component in the optical systems, find their parameters of Linear Canonical Transform. Then each component can be represented by a parameter matrix.
2. Then calculate the product of the parameter matrices, and we can obtain the parameters of Linear Canonical Transform for the whole system.

In **Fig. 4**, we see that if a monochromatic light with wavelength λ propagate through a free space with length d_0 two lenses with focus length f_1 and f_2, it can be represented by Linear Canonical Transform with parameters as

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -2\pi/\lambda f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda d_0/2\pi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2\pi/\lambda f_2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - d_0/f_2 & \lambda d_0/2\pi \\ \dfrac{2\pi}{\lambda} \left(\dfrac{d_0}{f_1 f_2} - \dfrac{1}{f_1} - \dfrac{1}{f_2} \right) & 1 - d_0/f_1 \end{bmatrix} \end{aligned} \tag{76}$$
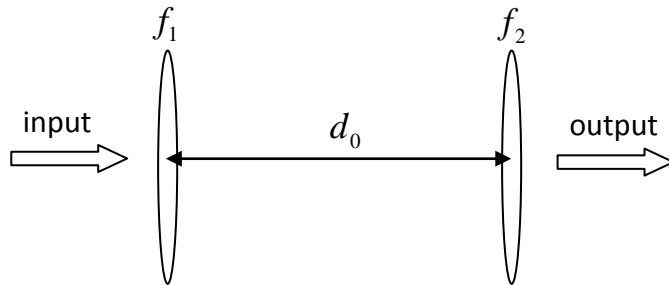
Fig. 4   the implementation of LCT with 2 cylinder lenses and 1 free space
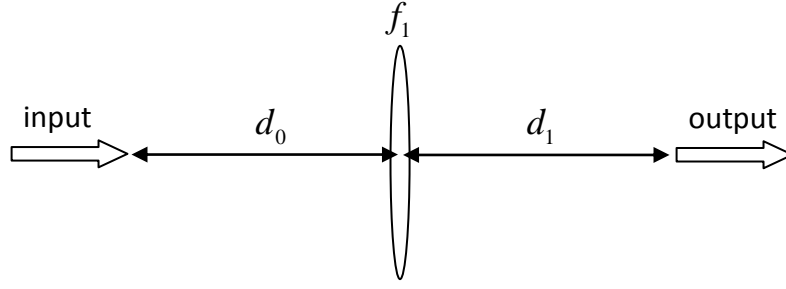
Fig. 5　　the implementation of LCT with 1 cylinder lens and 2 free spaces

5.1.3　　Implementing FrFT/LCT by Optical Systems

By the same concept as 5.1.2, we can also use optical system to implement Linear Canonical Transform. All the Linear Canonical Transform can be decomposed as the combination of the chirp multiplication and chirp convolution and we can decompose the parameter matrix into the following form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (d-1)/b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (a-1)/b & 1 \end{bmatrix} \quad \text{if } b \neq 0 \tag{77}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & (a-1)/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} 1 & (d-1)/c \\ 0 & 1 \end{bmatrix} \quad \text{if } c \neq 0 \tag{78}$$

Thus, for the case that $b \neq 0$, we can implement Linear Canonical Transform with two cylinder lenses and one free space as **Fig. 4**. Similarly, for the case that $c \neq 0$, we can implement Linear Canonical Transform with one cylinder lens and two free spaces as **Fig. 5**. And from Eq. (73)(75)(77)(78), we can find the focus length of lenses and the length of free spaces:

$$\text{for Fig. 4:} \quad f_1 = \frac{2\pi b}{\lambda(1-a)}, \quad d_0 = \frac{2\pi b}{\lambda}, \quad f_2 = \frac{2\pi b}{\lambda(1-d)} \tag{79}$$

$$\text{for Fig. 5:} \quad d_0 = \frac{2\pi(d-1)}{\lambda c}, \quad f_1 = -\frac{2\pi}{\lambda c}, \quad d_1 = \frac{2\pi(a-1)}{\lambda c} \tag{80}$$

Then, the relation between input and output will have the relation as the following equation

$$g_o(x) = \exp\left(-j\frac{2\pi L}{\lambda}\right)\left(O_F^{(a,b,c,d)}(g_i(x))\right) \tag{81}$$

where L is the length of the whole system.

5.2 Filtering and Noise Removal in Fractional Domains

In many signal processing applications, signals which we wish to recover are degraded by a known distortion and/or by noise. We may design some digital filter for noise removal. Fractional filter design and canonical filter design are discussed in many papers, they are the generalization of conventional filter design. Here we introduce the fractional filter design method; the canonical filter design is based on the same concept but only slightly different.

The conventional filter can be written as

$$x_o(t) = \int_{-\infty}^{+\infty} h(t-\tau) x_i(\tau) d\tau \tag{82}$$

where $x_i(t)$, $x_o(t)$ and $h(t)$ correspond to input signal, output signal and the impulse response of the filter. Eq. (82) can also be written in the frequency domain

$$x_o(t) = \frac{1}{\sqrt{2\pi}} IFT\left(FT\left(x_i(t)\right) \cdot H(w)\right), \quad \text{where } H(w) = FT\left(h(t)\right) \tag{83}$$

Fractional filter, as the generalization of conventional filter, is defined as

$$x_o(t) = \Im^{-\alpha}\left\{\Im^{-\alpha}\left\{x_i(t)\right\} \cdot H_\alpha(u)\right\}, \quad \text{where } H_\alpha(u) = \Im^\alpha\left\{h(t)\right\} \tag{84}$$

Due to the fact that performing the αth order Fractional Fourier Transform operation corresponds to rotating the Wigner Distribution by an angle $\phi = \alpha\dfrac{\pi}{2}$ in the clockwise direction, we can find the fractional domain that signal and noise do not have overlap. Then we can rotate the Wigner Distribution, that is, do the Fractional Fourier Transform, then filtering out the undesired noise. This is shown in **Fig. 6**. In **Fig. 6**, we see that for conventional filtering, to remove the noise in frequency domain is impossible. But we can rotate the Wigner Distribution and filter in the fractional
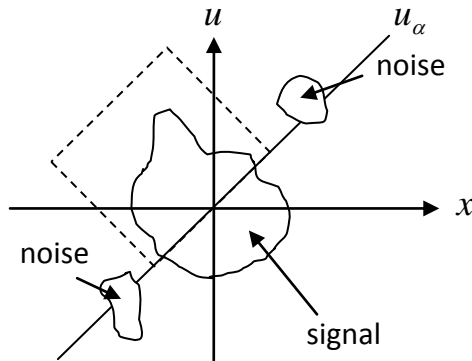
Fig. 6   Filtering in the fractional domain
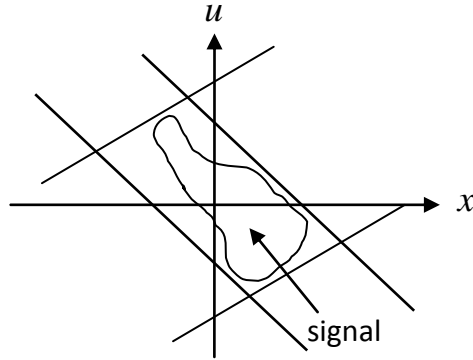
Fig. 7   Random noise removal by filtering in fractional domain

domain, then by choosing proper rotation angle and doing the same process iteratively, we may remove the noise easily. Recall that in **Fig. 1**, the Wigner Distribution of chirp function is a line delta. So it is easy to filter the noise in chirp form by the method just mentioned. By the same idea, we can remove random noise by applying fractional filters iteratively, see **Fig. 7**. After applying two times of fractional filters, by Eq. (28) the energy of noise is determined by the area circled by the four cutoff lines. Therefore, the smaller the area we circle, the smaller energy of the noise.

6   Conclusion and Future Works

In this paper, we've discussed the Fractional Fourier Transform. Linear Canonical Transform is also mentioned due to the high relativity to the Fractional Fourier Transform. As the generalization of Fourier Transform, Fractional Fourier Transform is a useful tool for signal processing. And since the flexibility of Fractional Fourier Transform is better than conventional Fourier Transform, many problems that cannot be solved well by conventional Fourier Transform are solved here.

Relation between other signal representations is one of the most important issues. Because of the simplicity of the relations, many applications are done with the Fractional Fourier Transform.

In chapter 4, we give the simplest method of implementing Fractional Fourier Transform in digital domain. However, there still have many other ways to implement Digital Fractional Fourier Transform.

Many applications are mentioned in chapter 5. We see that by the relation with Wigner Distribution, we may remove the undesired noise by doing filtering in fractional domain. This is accredited with the simple relation between Fractional Fourier Transform and Wigner Distribution.

In the future, we still have many research topics. We can try to find more efficient ways for implementing Digital Fractional Fourier Transform. And find other relation

between Fractional Fourier Transform and other signal representation. Moreover, try to find new applications of Fractional Fourier Transform since most of the applications now are optical applications.

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